Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 852406, 5 pages
doi:10.1155/2009/852406

Research Article

Bounds of Eigenvalues of $K_{3,3}$-Minor Free Graphs

Kun-Fu Fang

Faculty of Science, Huzhou Teachers College, Huzhou 313000, China

Correspondence should be addressed to Kun-Fu Fang, kffang@hutc.zj.cn

Received 17 February 2009; Accepted 11 May 2009

Recommended by Wing-Sum Cheung

The spectral radius $\rho(G)$ of a graph G is the largest eigenvalue of its adjacency matrix. Let $\lambda(G)$ be the smallest eigenvalue of G. In this paper, we have described the $K_{3,3}$-minor free graphs and showed that (A) let G be a simple graph with order $n \geq 7$. If G has no $K_{3,3}$-minor, then $\rho(G) \leq 1+\sqrt{3n-8}$. (B) Let G be a simple connected graph with order $n \geq 3$. If G has no $K_{3,3}$-minor, then $\lambda(G) \geq -\sqrt{2n-4}$, where equality holds if and only if G is isomorphic to $K_{2,n-2}$.

1. Introduction

In this paper, all graphs are finite undirected graphs without loops and multiple edges. Let G be a graph with $n = n(G)$ vertices, $m = m(G)$ edges, and minimum degree δ or $\delta(G)$. The spectral radius $\rho(G)$ of G is the largest eigenvalue of its adjacency matrix. Let $\lambda(G)$ be the smallest eigenvalue of G. The join $G\nabla H$ is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. A graph H is said to be a minor of G if H can be obtained from G by deleting edges, contracting edges, and deleting isolated vertices. A graph G is H-minor free if G has no H-minor.

Brualdi and Hoffman [1] showed that the spectral radius satisfies $\rho(G) \leq k-1$, where $m = k(k-1)/2$, with equality if and only if G is isomorphic to the disjoint union of the complete graph K_k and isolated vertices. Stanley [2] improved the above result. Hong et al. [3] showed that if G is a simple connected graph then $\rho \leq (\delta - 1 + \sqrt{(\delta+1)^2 + 4(2m-n\delta)})/2$ with equality if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n-1$. Hong [4] showed that if G is a K_5-minor free graph then (1) $\rho(G) \leq 1+\sqrt{3n-8}$, where equality holds if and only if G is isomorphic to $K_3\nabla(n-3)K_1$; (2) $\lambda(G) \geq -\sqrt{3n-9}$, where equality holds if and only if G is isomorphic to $K_{3,n-3}(n \geq 5)$.

In this paper, we have described the $K_{3,3}$-minor free graphs and obtained that

(a) let G be a simple graph with order $n \geq 7$. If G has no $K_{3,3}$-minor, then $\rho(G) \leq 1+\sqrt{3n-8}$;

(b) let G be a simple connected graph with order $n \geq 3$. If G has no $K_{3,3}$-minor, then $\lambda(G) \geq -\sqrt{2n-4}$, where equality holds if and only if G is isomorphic to $K_{2,n-2}$.

2. $K_{3,3}$-Minor Free Graphs

The intersection $G \cap H$ of G and H is the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$. Suppose G is a connected graph and S be a minimal separating vertex set of G. Then we can write $G = G_1 \cup G_2$, where G_1 and G_2 are connected and $G_1 \cap G_2 = G(S)$. Now suppose further that $G(S)$ is a complete graph. We say that G is a k-sum of G_1 and G_2, denoted by $G \equiv G_1 \oplus G_2$, if $|S| = k$. In particular, let $G_1 \oplus_2 G_2$ denote a 2–sum of G_1 and G_2. Moreover, if G_1 or G_2(say G_1) has a separating vertex set which induces a complete graph, then we can write $G_1 = G_3 \cup G_4$ such that G_3 and G_4 are connected and $G_3 \cap G_4$ is a complete subgraph of G. We proceed like this until none of the resulting subgraphs $G_1, G_2, \cdots, G_t$ has a complete separating subgraph. The graphs $G_1, G_2, \cdots, G_t$ are called the simplical summands of G. It is easy to show that the subgraphs $G_1, G_2, \cdots, G_t$ are independent of the order in which the decomposition is carried out (see [5]).

Theorem 2.1 (see [6], D. W. Hall; K. Wagner). *A graph has no $K_{3,3}$-minor if and only if it can be obtained by 0-, 1-, 2-summing starting from planar graphs and K_5.*

A graph G is said to be a edge-maximal H-minor free graph if G has no H-minor and G' has at least an H-minor, where G' is obtained from G by joining any two nonadjacent vertices of G. A graph G is called a maximal planar graph if the planarity will be not held by joining any two nonadjacent vertices of G.

Corollary 2.2. *If G is an edge maximal $K_{3,3}$-minor free graph then it can be obtained by 2-summing starting from K_5 and edge maximal planar graphs.*

Proof. This follows from Theorem 2.1. □

Lemma 2.3. *If G_1 and G_2 are two maximal planar graphs with order $n_1 \geq 3$ and $n_2 \geq 3$, respectively, then $G_1 \oplus_2 G_2$ is not a maximal planar graph.*

Proof. We denote a planar embedding of G_i by G_i still. Since G_i is a maximal planar graph, every face boundary in G_i is a 3-cycle. Hence the outside face boundary in $G_1 \oplus_2 G_2$ is a 4-cycle, this implies that the graph $G_1 \oplus_2 G_2$ is not maximal planar.

Further, we have the following results. □

Theorem 2.4. *If G is an edge-maximal $K_{3,3}$-minor free graph with $n \geq 3$ vertices then $G \cong G_0 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$, where $t = (n - n_0)/3$, G_0 is a maximal planar graph with order $2 \leq n_0 \leq n$. In particular,*

(1) *when $n_0 = 2$, $G \cong \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$, where $t = (n-2)/3$;*

(2) *when $n_0 = 3$, $G \cong K_3 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$, where $t = (n-3)/3$;*

(3) *when $n_0 = 4$, $G \cong K_4 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$, where $t = (n-4)/3$;*

(4) *when $n_0 = n$, $G \cong G_0$ is a maximal planar graph.*

Proof. Suppose that the graphs $G_1, G_2, \cdots, G_t (t \geq 1)$ are the simplical summands of G, namely $G \cong G_1 \oplus_2 G_2 \oplus_2 \cdots \oplus_2 G_t$. By Corollary 2.2, G_i is either a maximal planar graph or a K_5. By Lemma 2.3, there is at most a maximal planar graph in $G_i, 1 \leq i \leq t$. Hence we have $G \cong G_0 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$, where $t = (n - n_0)/3$, G_0 is a maximal planar graph with order $2 \leq n_0 \leq n$. □

Lemma 2.5 (see [7]). *Let G be a simple planar bipartite graph with $n \geq 3$ vertices and m edges. Then $m \leq 2n - 4$.*

Theorem 2.6. *Let G be a simple connected bipartite graph with $n \geq 3$ vertices and m edges. If G has no $K_{3,3}$-minor, then $m \leq 2n - 4$.*

Proof. Let H be a simple connected edge-maximal $K_{3,3}$-minor free graph with $n(H) = n(G)$ vertices and $m(H)$ edges. Suppose that the graphs $H_1, H_2, \cdots, H_t (t \geq 1)$ are the simplical summands of H. Then H_i is either a maximal planar graph or the graph K_5 by Corollary 2.2. Further, without loss generality, we may assume that G is a spanning subgraph of H. Let the graph G_i be the intersection of G and $H_i (1 \leq i \leq t)$. Then $n(G_i) = n(H_i)$ for $1 \leq i \leq t$. If $H_i \cong K_5$ then G_i is a subgraph of $K_{2,3}$, implies that $m(G_i) \leq 6 = 2n(G_i) - 4$. If H_i is a maximal planar graph then G_i is a simple planar bipartite graph, implies that $m(G_i) \leq 2n(G_i) - 4$ by Lemma 2.5. Next we prove this result by induction on t. For $t = 1$, $m = m(G) = m(G_1) \leq 2n(G_1) - 4 = 2n(G) - 4$. Now we assume it is true for $t = k$ and prove it for $t = k + 1$. Let $H' = H_1 \oplus H_2 \oplus \cdots \oplus H_k$ and $G' = G \cap H'$. Then $m(G') \leq 2n(G') - 4$ by the induction hypothesis.
$H = H' \oplus_2 H_{k+1}$. Hence $m(G) \leq m(G') + m(G_{k+1}) \leq 2(n(G') + n(G_{k+1}) - 2) - 4 = 2n(G) - 4$. □

3. Bounds of Eigenvalues of $K_{3,3}$-Minor Free Graphs

Lemma 3.1 (see [3]). *If G is a simple connected graph then $\rho \leq (\delta - 1 + \sqrt{(\delta + 1)^2 + 4(2m - n\delta)})/2$ with equality if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n - 1$.*

Lemma 3.2. *Let G be a simple connected graph with n vertices and m edges. If $\delta(G) \geq k$,then $\rho \leq (k - 1 + \sqrt{(k + 1)^2 + 4(2m - kn)})/2$, where equality holds if and only if $\delta(G) = k$ and G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n - 1$.*

Proof. Because when $n - 1 \leq m \leq n(n - 1)/2$ and $2m \geq xn$, $f(x) = (x - 1 + \sqrt{(x + 1)^2 + 4(2m - nx)})/2$ is a decreasing function of x for $1 \leq x \leq n - 1$, this follows from Lemma 3.1. □

Lemma 3.3. *Let G_0 be a maximal planar graph with order n_0, and let G be a graph with n vertices and m edges.*

(1) *If $G \cong \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$ and $n \geq 5$, where $t = (n - 2)/3$, then $m = 3n - 5, \delta(G) = 4$.*

(2) *If $G \cong K_3 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$ and $n \geq 6$, where $t = (n - 3)/3$, then $m = 3n - 6$, $\delta(G) = 2$.*

(3) *If $G \cong G_0 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$ and $n \geq n_0 \geq 4$, where $t = (n - n_0)/3$, then $m = 3n - 6$, $\delta(G) \geq 3$.*

Proof. Applying the properties of the maximal planar graphs, this follows by calculating. □

Lemma 3.4. *Let G_0 be a maximal planar graph with order n_0, and let G be a graph with n vertices.*

(1) *If $G \cong \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$ and $n \geq 5$, where $t = n - 2/3$, then $\rho(G) \leq (3 + \sqrt{8n-15})/2$.*

(2) *If $G \cong K_3 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$ and $n \geq 6$, where $t = n - 3/3$, then $\rho(G) < (3 + \sqrt{8n+1})/2$.*

(3) *If $G \cong G_0 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$ and $n \geq n_0 \geq 4$, where $t = n - n_0/3$, then $\rho(G) \leq 1 + \sqrt{3n-8}$.*

Proof. It follows that (1) and (3) are true by Lemma 3.2 and 5(1)(3). Next we prove that (2) is true too.

Let G^* be a graph obtained from G by expanding K_3 (in the simplcal summands of G) to K_5, such that G^* can be obtained by 2-summing K_5, namely, $G^* \cong \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t+1}$.

This implies that $\rho(G^*) \leq (3 + \sqrt{8n^* - 15})/2$ by (1). Also we have $n^* = n(G^*) = n(G) + 2 = n + 2$, so $\rho(G) < \rho(G^*) \leq (3 + \sqrt{8n+1})/2$. □

Theorem 3.5. *Let G be a simple graph with order $n \geq 7$. If G has no $K_{3,3}$-minor, then $\rho(G) \leq 1 + \sqrt{3n-8}$.*

Proof. Since when adding an edge in G the spectral radius $\rho(G)$ is strict increasing, we consider the edge-maximal $K_{3,3}$-minor free graph only. Next we may assume that G is an edge-maximal $K_{3,3}$-minor free graph.

By Theorem 2.4 and Lemma 3.4, when $n \geq 4$, $\rho(G) \leq \max\{(1 + \sqrt{3n-8}), (3 + (\sqrt{8n-15})/2),\ 3 + (\sqrt{8n+1}/2)\}$.

When $n \geq 14$, $1 + \sqrt{3n-8} > max\{3 + (\sqrt{8n-15})/2,\ (3 + \sqrt{8n+1})/2\}$.

When $7 \leq n \leq 13$, we have $\rho(G) \leq \rho(G_0 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}) \leq 1 + \sqrt{3n-8}$ by calculating directly, where $t = (n - n_0)/3$, G_0 is a maximal planar graph with order $2 \leq n_0 \leq n$ (see Theorem 2.4).

Therefore when $n \geq 7$, $\rho(G) \leq 1 + \sqrt{3n-8}$. □

Remark 3.6. In Theorem 3.5, the equality holds only if $n = 8$, for the others, the upper bounds of $\rho(G)$ are not sharp. We conjecture that the best bound of $\rho(G)$ is $(3 + \sqrt{8n-15})/2$ still.

Lemma 3.7 (see [7]). *If G is a simple connected graph with n vertices, then there exists a connected bipartite subgraph H of G such that $\lambda(G) \geq \lambda(H)$ with equality holding if and only if $G \cong H$.*

Lemma 3.8 (see [7]). *If G is a connected bipartite graph with n vertices and m edges, then $\lambda(G) \geq -\sqrt{m}$, where equality holds if and only if G is a complete bipartite graph.*

Theorem 3.9. *Let G be a simple connected graph with $n \geq 3$ vertices. If G has no $K_{3,3}$-minor, then $\lambda(G) \geq -\sqrt{2n-4}$, where equality holds if and only if G is isomorphic to $K_{2,n-2}$.*

Proof. This follows from Lemmas 3.7, 3.8 and Theorem 2.6. □

Acknowledgments

The author wishes to express his thanks to the referee for valuable comments which led to an improved version of the paper. Work supported by NNSF of China (no. 10671074) and NSF of Zhejian Province (no. Y7080364).

References

[1] R. A. Brualdi and A. J. Hoffman, "On the spectral radius of $(0,1)$-matrices," *Linear Algebra and Its Applications*, vol. 65, pp. 133–146, 1985.

[2] R. P. Stanley, "A bound on the spectral radius of graphs with e edges," *Linear Algebra and Its Applications*, vol. 87, pp. 267–269, 1987.

[3] Y. Hong, J.-L. Shu, and K. F. Fang, "A sharp upper bound of the spectral radius of graphs," *Journal of Combinatorial Theory, Series B*, vol. 81, no. 2, pp. 177–183, 2001.

[4] Y. Hong, "Tree-width, clique-minors, and eigenvalues," *Discrete Mathematics*, vol. 274, no. 1–3, pp. 281–287, 2004.

[5] C. Thomassen, "Embeddings and minors," in *Handbook of Combinatorics, Vol. 1, 2*, R. Graham, M. Grotschel, and L. Lovasz, Eds., pp. 301–349, Elsevier, Amsterdam, The Netherlands, 1995.

[6] J. A. Bondy and U. S. R. Murty, *Graph Theory*, vol. 244 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2008.

[7] Y. Hong and J.-L. Shu, "Sharp lower bounds of the least eigenvalue of planar graphs," *Linear Algebra and Its Applications*, vol. 296, no. 1–3, pp. 227–232, 1999.